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From Gödel quaternions to non-linear sigma models

D Lambert† and J Rembielinski‡

† Institute de Physique Théorique, Université Catholique de Louvain, B-1348 Louvain-La-Neuve, Belgium

‡ Institute of Physics, University of Lodz, PL-90-136 Lodz, Poland

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Abstract. From the ring of Gödel quaternions, we construct new quaternionic groups and a pseudo-Hopf fibration. These can be used to study sigma models valued in a four-dimensional hyperboloid. Particular solutions of these models are also given.

1. Introduction

In a previous work (Lambert and Piette 1988) we have used the ring of hyperbolic complex numbers in order to generate solutions of two-dimensional Minkowskian sigma models on particular non-compact manifolds. In the present paper, we will show that the ring of hyperbolic quaternions (also called Gödel quaternions) can be introduced in order to describe sigma models defined on pseudo-Riemannian manifolds and valued in the four-dimensional single-sheeted hyperboloid $SO(3, 2)/SO(2, 2)$.

Gödel was the first to point out the physical usefulness of hyperbolic quaternions when he gave his famous solution of Einstein's field equation of gravitation (Gödel 1949). In the same spirit, Ozsvath used these quaternions in studying particular classes of dust-filled universes (Ozsvath 1970). Looking for the pseudo-meron solution of Yang-Mills equations on the hyperboloid $SO(2, 2)/SO(1, 2)$, Hogan used the relation between the Lie group $SL(2, \mathbb{R})$ and the set of Gödel quaternions of unit norm (Hogan 1984). In the framework of a non-symmetric theory of gravitation, Gödel quaternions were also introduced by Moffat to represent geometrical quantities and spinors (Moffat 1984). Finally, these quaternions have been used to generate physically relevant non-bijective canonical transformations (Lambert and Kibler 1988). For the sake of completeness, it is worth noting that the ring of Gödel quaternions was extensively studied in the framework of Clifford algebras and in the theory of real 2-spinors (Ilamed and Salingaros 1981).

The paper is organised as follows. In § 2, we recall the definition and properties of the ring $\mathbb{Q}$ of Gödel quaternions. In § 3, we introduce the group $GL(n, \mathbb{Q})$ and study some of its subgroups. We also consider the relations between these subgroups and the usual groups $O(2n)$, $O(n, n)$ and $Sp(n, \mathbb{R})$. Using the properties of $\mathbb{Q}$, we construct, in § 4, a pseudo-Hopf fibration and show that it defines a harmonic map between the hyperboloids $SO(4, 4)/SO(3, 4)$ and $SO(3, 2)/SO(2, 2)$. In § 5, we define sigma models valued in the hyperboloid $SO(3, 2)/SO(2, 2)$. Using the results of §§ 3 and 4 we give equivalent forms of the Lagrangian density of these models. From this, it is possible to understand the relation between these models and $SL(2, \mathbb{R})$ gauge field theories. In § 6, finally, we give examples of particular solutions of these sigma models.

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2. Gödel quaternions

Let e_0, e_1, e_2 and e_3 be a base of a four-dimensional real vector space. We define the Gödel quaternion algebra $\mathbb{Q}$ introducing the non-commutative multiplication given by the following requirements:

$$e_0 e_k = e_k = e_k e_0 \qquad (e_0)^2 = e_0$$

$$e_i e_k = (-1)^j \varepsilon_{ikj} e_j - (-1)^k \delta_{ik} e_0 \qquad i, j, k = 1, 2, 3.$$

Let q_0, q_1, q_2 and q_3 be arbitrary real numbers. The vector

$$q = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 \tag{2.1}$$

is called a Gödel quaternion.

It is also possible to define another basis of $\mathbb{Q}$ using the following vectors:

$$e_{11} = (e_0 + e_3)/2 \qquad e_{22} = (e_0 - e_3)/2 \tag{2.2}$$

$$e_{12} = (e_1 + e_2)/2 \qquad e_{21} = (e_1 - e_2)/2. \tag{2.3}$$

These satisfy the multiplication law

$$e_{ik} e_{rs} = \delta_{kr} e_{is} \qquad i, k, r, s = 1, 2. \tag{2.4}$$

Now, every Gödel quaternion q can be expressed as follows:

$$q = \sum_{i,k=1,2} q_{ik} e_{ik} \tag{2.5}$$

where q_{ik} is an arbitrary real number.

The proper subalgebras of $\mathbb{Q}$ are the algebra $\mathbb{R}$ of real numbers generated by e_0, the algebra $\mathbb{C}$ of usual complex numbers generated by e_0 and e_2, with $(e_2)^2 = -e_0$, and the algebra $\Omega(e_1)$ (respectively $\Omega(e_3)$) of hyperbolic complex (or double) numbers (Yaglom 1968) generated by e_0 and e_1, with $(e_1)^2 = e_0$ (respectively generated by e_0 and e_3, with $(e_3)^2 = e_0$). Obviously, $\Omega(e_1)$ is isomorphic to $\Omega(e_3)$. Then, in the following, the algebra of hyperbolic complex numbers will be simply denoted by Ω.

Let A be an algebra. We recall that the map $j: A \to A$ is an involution (respectively an anti-involution) of A if and only if $j^2(h) = j(h)$ and $j(gh) = j(g)j(h)$ (respectively $j(gh) = j(h)j(g)$) for every g and h in A.

Let us now introduce the $\mathbb{Q}$-conjugated element $\bar{q}$ of q by the following expression:

$$\bar{q} = q_0 e_0 - q_1 e_1 - q_2 e_2 - q_3 e_3 = q_{22} e_{11} + q_{11} e_{22} - q_{12} e_{12} - q_{21} e_{21}. \tag{2.6}$$

The map $q \to \bar{q}$ defines an anti-involution of $\mathbb{Q}$. Furthermore, if $q = \bar{q}$ then q is simply a real number. It is also possible to define the Ω-conjugated element $\tilde{q}$ of q by

$$\tilde{q} = q_0 e_0 - q_1 e_1 + q_2 e_2 - q_3 e_3 = q_{22} e_{11} + q_{11} e_{22} - q_{21} e_{12} - q_{12} e_{21}. \tag{2.7}$$

The map $q \to \tilde{q}$ defines an involution of $\mathbb{Q}$. Each element q of $\mathbb{Q}$ can be written as follows:

$$q = a e_0 + b e_2$$

where $a = q_0 e_0 + q_3 e_3$ and $b = q_2 e_0 + q_1 e_3$ are hyperbolic complex numbers. In this representation, we get

$$\tilde{q} = \tilde{a} e_0 + \tilde{b} e_2.$$

Then the condition $q = \tilde{q}$ means that q is an element of $\mathbb{C}$. Finally, let us define the $\mathbb{C}$-conjugated element q^* of q as follows:

$$q^* = q_0 e_0 - q_1 e_1 - q_2 e_2 + q_3 e_3 = q_{11} e_{11} + q_{22} e_{22} - q_{12} e_{12} - q_{21} e_{21}. \tag{2.8}$$

The map $q \to q^*$ defines an involution of $\mathbb{Q}$. Each element $\mathbb{Q}$ can be expressed by the following equation:

$$q = ce_0 + de_3$$

where $c = q_0e_0 + q_2e_2$ and $d = q_3e_0 - q_1e_2$ are usual complex numbers. In this representation we get

$$q^* = c^*e_0 + d^*e_3.$$

The constraint $q = q^*$ means that q is a hyperbolic complex number. The $\mathbb{Q}$ conjugation allows us to define the square of the norm $|q|^2$ of every element q of $\mathbb{Q}$ as follows:

$$|q|^2 = \bar{q}q = q\bar{q} = (q_0)^2 - (q_1)^2 + (q_2)^2 - (q_3)^2.$$

We note that $\mathbb{Q}$ involves a set $\mathbb{Q}_0$ of zero divisors (i.e. elements such that $|q|^2 = 0$). Therefore, $\mathbb{Q}$ is neither a division algebra nor a field (as the usual quaternion algebra) but a non-commutative ring. For every q and r in $\mathbb{Q}$ the following property:

$$|qr|^2 = |q|^2|r|^2$$

holds. This follows from the non-compact generalisation of the well known theorem of Hurwitz (Lambert and Kibler 1988) about composition algebras. The subset of $\mathbb{Q}$ with generic element q such that $|q|^2 = 1$ is isomorphic to the three-dimensional hyperboloid $H^3(2, 2)$ where $H^n(u, v)$ denotes the n-dimensional manifold of $\mathbb{R}^{n+1}$ defined by the equation

$$\sum_{i=1,\dots,u} (q_i)^2 - \sum_{i=u+1,\dots,u+v} (q_i)^2 = 1 \qquad u + v = n.$$

From the isomorphism $H^3(2, 2) \simeq \mathrm{SL}(2, \mathbb{R})$, it follows that Gödel quaternions of unit norm provide a representation of the group $\mathrm{SL}(2, \mathbb{R})$ (Gödel 1949, Ozsvath 1970, Hogan 1984). For those preferring not to work with the generators e_0, e_1, e_2 and e_3, it is possible to construct a 2×2 matrix representation of $\mathbb{Q}$ over $\mathbb{R}$. This is achieved by using the following identifications:

$$e_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad e_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad e_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad e_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

From this, an arbitrary Gödel quaternion q of $\mathbb{Q}$ can be identified with the 2×2 real matrix:

$$\begin{bmatrix} q_0 + q_3 & q_1 + q_2 \\ q_1 - q_2 & q_0 - q_3 \end{bmatrix}.$$

This shows in fact that $\mathbb{Q}$ is isomorphic to the ring $\mathbb{R}(2)$ of real 2×2 matrices. Furthermore, it is worth recalling that $\mathbb{Q}$ can be viewed as the Clifford algebra $C(2, 0) \simeq C(1, 1)$, which is precisely isomorphic to $\mathbb{R}(2)$ (Porteous 1969).

3. The group $\mathrm{GL}(n, \mathbb{Q})$ and its subgroups

Let $\mathbb{Q}^n$ be the set of all $n\times 1$ matrices with elements in $\mathbb{Q}$. Due to the fact that $\mathbb{Q}$ is a ring and not a field, $\mathbb{Q}^n$ is not a vector space but a $\mathbb{Q}$ module. We now define the set

$\mathrm{GL}(n, \mathbb{Q})$ of all invertible and $\mathbb{R}$-linear mappings: $\mathbb{Q}^n \to \mathbb{Q}^n$. An arbitrary element F of $\mathbb{Q}^n$ can be written as follows:

$$F = \begin{bmatrix} F^1 \\ \vdots \\ F^n \end{bmatrix} = \sum_{\mu=0,\ldots,3} f^\mu e_\mu = \sum_{i,k=1,2} f^{ik} e_{ik}$$

where f^μ and f^{ik} are real $n \times 1$ matrices defined by

$$f^\mu = \begin{bmatrix} F^1_\mu \\ \vdots \\ F^n_\mu \end{bmatrix} \qquad f^{ik} = \begin{bmatrix} F^1_{ik} \\ \vdots \\ F^n_{ik} \end{bmatrix}$$

and

$$F^j = \sum_{\mu=0,\ldots,3} F^j_\mu e_\mu = \sum_{i,k=1,2} F^j_{ik} e_{ik}$$

according to (2.1) and (2.5). From this, it follows that $\mathrm{GL}(n, \mathbb{Q})$ is the group of all invertible $n \times n$ matrices over $\mathbb{Q}$. Let A and B be matrices of $\mathrm{GL}(n, \mathbb{Q})$. According to (2.1) and (2.5), we get

$$\begin{aligned} A &= \sum_{\mu=0,\ldots,3} A^\mu e_\mu = \sum_{i,k=1,2} A^{ik} e_{ik} \\ B &= \sum_{\mu=0,\ldots,3} B^\mu e_\mu = \sum_{i,k=1,2} B^{ik} e_{ik} \end{aligned} \tag{3.1}$$

where A_μ, B_μ and A^{ik}, B^{ik} are $n \times n$ real matrices.

Let us now consider the map $J: \mathrm{GL}(n, \mathbb{Q}) \to \mathrm{GL}(2n, \mathbb{R})$ defined by the following expression:

$$J(A) = \begin{bmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{bmatrix}. \tag{3.2}$$

Using (3.1) and the multiplication law (2.4) we check that

$$J(AB) = J(A)J(B). \tag{3.3}$$

Thus, J defines a group homomorphism between $\mathrm{GL}(n, \mathbb{Q})$ and $\mathrm{GL}(2n, \mathbb{R})$. Hence, it is possible to show that J is in fact an isomorphism. Then we have the following identification:

$$\mathrm{GL}(n, \mathbb{Q}) \simeq \mathrm{GL}(2n, \mathbb{R}).$$

Let us now exhibit some relevant subgroups of $\mathrm{GL}(n, \mathbb{Q})$. It is worth recalling that if A belongs to $\mathrm{GL}(n, \mathbb{Q})$, then, according to § 2, the following constraints:

$$\bar{A} = A \qquad \tilde{A} = A \qquad A^* = A$$

mean that A is a $n \times n$ real, complex and hyperbolic complex matrix, respectively. We now introduce particular subsets of $\mathrm{GL}(n, \mathbb{Q})$ given by

$$\mathrm{GL}(n, \bar{\mathbb{Q}}) = \{A \in \mathrm{GL}(n, \mathbb{Q}) : \bar{A} = A\} \tag{3.4}$$

$$\mathrm{GL}(n, \tilde{\mathbb{Q}}) = \{A \in \mathrm{GL}(n, \mathbb{Q}) : \tilde{A} = A\} \tag{3.5}$$

$$\mathrm{GL}(n, \mathbb{Q}^*) = \{A \in \mathrm{GL}(n, \mathbb{Q}) : A^* = A\}. \tag{3.6}$$

It is now straightforward to check that these subsets are in fact subgroups. Furthermore, we have the following identifications:

$$\mathrm{GL}(n,\bar{\mathbb{Q}}) \simeq \mathrm{GL}(n,\mathbb{R}) \qquad \mathrm{GL}(n,\tilde{\mathbb{Q}}) \simeq \mathrm{GL}(n,\mathbb{C}) \qquad \mathrm{GL}(n,\mathbb{Q}^*) \simeq \mathrm{GL}(n,\Omega). \tag{3.7}$$

Then it is easy to see that the following relation:

$$\mathrm{GL}(n,\bar{\mathbb{Q}}) \simeq \mathrm{GL}(n,\tilde{\mathbb{Q}}) \cap \mathrm{GL}(n,\mathbb{Q}^*)$$

holds.

In the following, we will use the map τ defined for every matrix A of $\mathrm{GL}(n,\mathbb{Q})$ by the following equation:

$$A^\tau = (\tilde{\bar{A}})^{\mathrm{T}} \tag{3.8}$$

where T denotes the usual matrix transposition. Using the isomorphism J (3.2) and the definition (3.8), we get

$$J(A^\tau) = J(A)^{\mathrm{T}}. \tag{3.9}$$

Then we are able to prove that τ is an anti-involution of $\mathrm{GL}(n,\mathbb{Q})$. We have to check the condition

$$(AB)^\tau = B^\tau A^\tau$$

for every matrix A and B in $\mathrm{GL}(n,\mathbb{Q})$. Using (3.3) and (3.9), we get

$$J((AB)^\tau) = J(B)^{\mathrm{T}} J(A)^{\mathrm{T}} = J(B^\tau) J(A^\tau)$$

and

$$J(B^\tau A^\tau) = J(B^\tau) J(A^\tau).$$

According to the fact that J is a group isomorphism, we can complete the proof. Let us define the matrix E_μ of $\mathrm{GL}(n,\mathbb{Q})$ by the equation

$$E_\mu = e_\mu \mathbb{1}_n \qquad \mu = 0, 1, 2, 3$$

where $\mathbb{1}_n$ denotes the $n \times n$ unit matrix. We can easily check the following equations:

$$J(E_0) = \begin{bmatrix} \mathbb{1}_n & \\ & \mathbb{1}_n \end{bmatrix} \qquad J(E_3) = \begin{bmatrix} \mathbb{1}_n & \\ & -\mathbb{1}_n \end{bmatrix} \tag{3.10}$$

$$J(E_1) = \begin{bmatrix} & \mathbb{1}_n \\ \mathbb{1}_n & \end{bmatrix} \qquad J(E_2) = \begin{bmatrix} & \mathbb{1}_n \\ -\mathbb{1}_n & \end{bmatrix}. \tag{3.11}$$

These follow from (2.2) and (2.3) which give

$$e_0 = e_{11} + e_{22} \qquad e_3 = e_{11} - e_{22}$$

$$e_1 = e_{12} + e_{21} \qquad e_1 = e_{12} - e_{21}$$

and from the definition of the isomorphism J. The anti-involution τ and the matrices E_μ can be used to define new subgroups of $\mathrm{GL}(n,\mathbb{Q})$. Let us first consider the subsets $\mathrm{O}_\mu(n,\mathbb{Q})$ of $\mathrm{GL}(n,\mathbb{Q})$, with $\mu = 0$, 1, 2 and 3, defined by

$$\mathrm{O}_\mu(n,\mathbb{Q}) = \{A \in \mathrm{GL}(n,\mathbb{Q}) \colon A^\tau E_\mu A = E_\mu\}. \tag{3.12}$$

These subsets, which happen to be subgroups of $GL(n, \mathbb{Q})$, will be called orthogonal groups over $\mathbb{Q}$. From (3.3), (3.9) and (3.11), we get the following identifications:

$$O_0(n, \mathbb{Q}) \simeq O(2n) \tag{3.13}$$

$$O_1(n, \mathbb{Q}) \simeq O_3(n, \mathbb{Q}) \simeq O(n, n) \tag{3.14}$$

$$O_2(n, \mathbb{Q}) \simeq Sp(n, \mathbb{R}). \tag{3.15}$$

Referring to (3.12) and (3.7), we also check that

$$O_\mu(n, \mathbb{Q}) \cap GL(n, \bar{\mathbb{Q}}) \simeq O(n) \qquad \mu = 0, 1, 2, 3.$$

The group $GL(n, \tilde{\mathbb{Q}})$ leads to the following equations:

$$O_0(n, \mathbb{Q}) \cap GL(n, \tilde{\mathbb{Q}}) \simeq O_2(n, \mathbb{Q}) \cap GL(n, \tilde{\mathbb{Q}}) \simeq U(n)$$

$$O_1(n, \mathbb{Q}) \cap GL(n, \tilde{\mathbb{Q}}) \simeq O_3(n, \mathbb{Q}) \cap GL(n, \tilde{\mathbb{Q}}) \simeq O(n, \mathbb{C}).$$

Finally, the group $GL(n, \mathbb{Q}^*)$ gives rise to the following identifications:

$$O_0(n, \mathbb{Q}) \cap GL(n, \mathbb{Q}^*) \simeq O_3(n, \mathbb{Q}) \cap GL(n, \mathbb{Q}^*) \simeq O(n, \Omega)$$

$$O_1(n, \mathbb{Q}) \cap GL(n, \mathbb{Q}^*) \simeq O_2(n, \mathbb{Q}) \cap GL(n, \mathbb{Q}^*) \simeq U(n, \Omega).$$

In this last case, we find hyperbolic complex orthogonal and unitary groups $O(n, \Omega)$ and $U(n, \Omega)$ which were introduced by Zhong (1984, 1985). According to Zhong (1985), we can get the following isomorphisms:

$$O(n, \Omega) \simeq O(n) \times O(n) \qquad U(n, \Omega) \simeq GL(n, \mathbb{R}).$$

It is also possible to introduce pseudo-orthogonal groups over $\mathbb{Q}$. For this purpose, let us consider the matrix D_{pq} of $GL(n, \mathbb{Q})$ defined by

$$D_{pq} = K_{pq} e_0 + (\mathbb{1}_n - K_{pq}) e_3 \qquad p + q = 2n$$

where K_{pq} is the $n \times n$ real matrix given by

$$K_{pq} = \begin{bmatrix} \mathbb{1}_{p-n} & \\ & \mathbb{O}_q \end{bmatrix} \qquad p > q$$

$$K_{pq} = \mathbb{O}_n \qquad p = q$$

with $\mathbb{O}_q$ denoting the $q \times q$ null matrix. Through the map J, the matrix D_{pq} becomes

$$J(D_{pq}) = \begin{bmatrix} \mathbb{1}_p & \\ & -\mathbb{1}_q \end{bmatrix} \qquad p > q$$

$$J(D_{pq}) = \begin{bmatrix} \mathbb{1}_n & \\ & -\mathbb{1}_n \end{bmatrix} \qquad p = q.$$

This can be seen by writing D_{pq} in the basis (2.2) and (2.3). More precisely, we have

$$D_{pq} = \begin{bmatrix} \mathbb{1}_{p-n} & \\ & \mathbb{1}_q \end{bmatrix} e_{11} + \begin{bmatrix} \mathbb{1}_{p-n} & \\ & -\mathbb{1}_q \end{bmatrix} e_{22} \qquad p > q$$

$$D_{pq} = \mathbb{1}_n (e_{11} - e_{22}) \qquad p = q.$$

Now let us consider the subset $O(p, q; \mathbb{Q})$ of $GL(n, \mathbb{Q})$ defined by

$$O(p, q; \mathbb{Q}) = \{A \in GL(n, \mathbb{Q}) : A^T D_{pq} A = D_{pq}\} \qquad p + q = 2n \qquad p \geqslant q.$$

We easily check that this subset is a subgroup of $\mathrm{GL}(n, \mathbb{Q})$. Furthermore, after a straightforward computation, we get the following isomorphism:

$$\mathrm{O}(p, q; \mathbb{Q}) \simeq \mathrm{O}(p, q) \qquad p \geqslant q.$$

Finally, we have to investigate unitary-like groups over $\mathbb{Q}$. For this purpose, we introduce two new conjugations on $\mathrm{GL}(n, \mathbb{Q})$. We define, for every matrix A of $\mathrm{GL}(n, \mathbb{Q})$, the matrices A^+ and $A^{\#}$ given by

$$A^+ = (A^\tau)^* \qquad A^{\#} = (\tilde{A}^\tau) = (\bar{A})^{\mathrm{T}}. \tag{3.16}$$

From (2.6)-(2.8), it comes that

$$A^+ = e_3 A^\tau e_3 \qquad A^{\#} = -e_2 A^\tau e_2. \tag{3.17}$$

We now introduce, for each $\mu = 0$, 1, 2 and 3, the subsets

$$\begin{aligned} \mathrm{U}_\mu^{(+)}(n, \mathbb{Q}) &= \{A \in \mathrm{GL}(n, \mathbb{Q}) \colon A^+ E_\mu A = E_\mu\} \\ \mathrm{U}_\mu^{(\#)}(n, \mathbb{Q}) &= \{A \in \mathrm{GL}(n, \mathbb{Q}) \colon A^{\#} E_\mu A = E_\mu\}. \end{aligned} \tag{3.18}$$

These subsets, which happen to be subgroups of $\mathrm{GL}(n, \mathbb{Q})$, will be called unitary groups over $\mathbb{Q}$. From (3.17), we can easily check that the equation

$$A^+ E_\mu A = E_\mu$$

implies the following identity:

$$A^\tau (e_3 e_\mu) A = (e_3 e_\mu) \mathbb{1}_n.$$

According to the multiplication law of Gödel quaternions we get

$$e_3 e_\mu = e_{3-\mu}.$$

Then, from (3.12) we get the following identification:

$$\mathrm{U}_\mu^{(+)}(n, \mathbb{Q}) \simeq \mathrm{O}_{3-\mu}(n, \mathbb{Q}). \tag{3.19}$$

From (3.17), it follows that the condition

$$A^{\#} E_\mu A = E_\mu$$

can be written

$$A^\tau (e_2 e_\mu) A = (e_2 e_\mu) \mathbb{1}_n.$$

Using the multiplication table of Gödel quaternions and definition (3.12), we get the following identifications:

$$\mathrm{U}_0^{(\#)}(n, \mathbb{Q}) \simeq \mathrm{O}_2(n, \mathbb{Q}) \qquad \mathrm{U}_1^{(\#)} \simeq \mathrm{O}_2(n, \mathbb{Q}) \tag{3.20}$$

$$\mathrm{U}_2^{(\#)}(n, \mathbb{Q}) \simeq \mathrm{O}_0(n, \mathbb{Q}) \qquad \mathrm{U}_3^{(\#)} \simeq \mathrm{O}_1(n, \mathbb{Q}). \tag{3.21}$$

All the relations (3.19)-(3.21) show that the unitary groups over $\mathbb{Q}$ are, in fact, orthogonal groups over $\mathbb{Q}$. The group $\mathrm{U}_0^{(\#)}(n, \mathbb{Q})$ is the analogue of the usual unitary group $\mathrm{U}(n)$. This can be seen by identifying e_0 with the real unit and introducing (3.16) in the definition (3.18).

In figure 1, we summarise results of § 3. Lines denote identifications (isomorphisms) and arrows denote canonical inclusions.

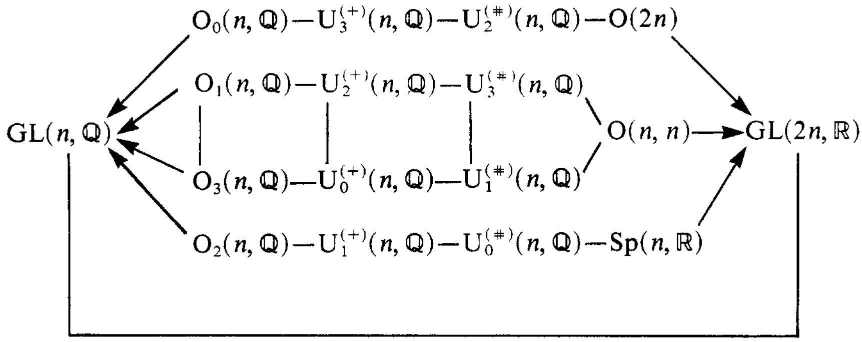

Figure 1. The group $GL(n, \mathbb{Q})$ and its subgroups.

4. A pseudo-Hopf fibration

Let q and r be arbitrary Gödel quaternions defined by

$$q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3 \qquad r = r_0e_0 + r_1e_1 + r_2e_2 + r_3e_3.$$

We define the map $F: \mathbb{Q} \times \mathbb{Q} \to \mathbb{R} \times \mathbb{Q}$ as follows:

$$F(q, r) = (|q|^2 - |r|^2, 2q\bar{r}). \tag{4.1}$$

Let s be a Gödel quaternion such that

$$|s|^2 = 1. \tag{4.2}$$

Then we check that the following property:

$$F(qs, rs) = F(q, r) \tag{4.3}$$

holds for all q and r in $\mathbb{Q}$. From (4.2), (4.3) and § 2, we exhibit the $O_2(1, \mathbb{Q}) \simeq SL(2, \mathbb{R})$ fibre of F. The map F induces a map $f: \mathbb{R}^8 \to \mathbb{R}^5$ defined by $X = f(U)$, where U is the eight-dimensional real vector given by $U^T = (q_0, q_1, q_2, q_3, r_0, r_1, r_2, r_3)$ and X is the five-dimensional real vector such that $X^T = (X_1, X_2, X_3, X_4, X_5)$ and

$$\begin{aligned}
X_1 &= (q_0)^2 - (q_1)^2 + (q_2)^2 - (q_3)^2 - (r_0)^2 + (r_1)^2 - (r_2)^2 + (r_3)^2 \\
X_2 &= 2(q_0r_0 - q_1r_1 + q_2r_2 - q_3r_3) \\
X_3 &= 2(q_1r_0 - q_0r_1 + q_2r_3 - q_3r_2) \\
X_4 &= 2(q_2r_0 - q_0r_2 + q_1r_3 - q_3r_1) \\
X_5 &= 2(q_3r_0 - q_0r_3 + q_1r_2 - q_2r_1).
\end{aligned} \tag{4.4}$$

By a straightforward computation, we get

$$X^T hX = (U^T NU)^2 \tag{4.5}$$

where h is the 5×5 matrix defined by $h = \mathrm{diag}(1, 1, -1, 1, -1)$ and N is the 8×8 matrix given by $N = \mathrm{diag}(1, -1, 1, -1, -1, 1, -1)$. If we restrict f to the hyperboloid $H^7(4, 4)$ we get, according to (4.5), the map $\hat{f}: H^7(4, 4) \to H^4(3, 2)$. From (4.3) we easily see that $\hat{f}$ is a fibration of fibre $SL(2, \mathbb{R})$. In a previous work (Lambert and Kibler 1988), we have introduced this fibration from a different point of view and we have called it a pseudo-Hopf fibration. The hyperboloid $H^7(4, 4)$ (respectively $H^4(3, 2)$) is homeomorphic to $\mathbb{R}^4 \times S^3$ (respectively $\mathbb{R}^2 \times S^2$). Then, because of the well known result: $\pi_3(S^2) = \mathbb{Z}$, we see that $\hat{f}$ is a non-trivial fibration (Steenrod 1974).

Let (M_1, g_1) (respectively (M_2, g_2)) be a pseudo-Riemannian manifold of dimension m_1 (respectively m_2) endowed with the metric g_1 (respectively g_2). Then, a smooth map

$$Y:(M_1, g_1)\to(M_2, g_2): x=(x^1, \ldots, x^{m_1})\to Y(x)=(Y^1, \ldots, Y^{m_2}) \tag{4.6}$$

is said to be a harmonic map (Eells and Lemaire 1978) if and only if

$$g_1^{\alpha\beta}\left(\frac{\partial^2 Y^l}{\partial x^\alpha\,\partial x^\beta}-{}^{(1)}\Gamma^\gamma_{\alpha\beta}\frac{\partial Y^l}{\partial x^\gamma}+{}^{(2)}\Gamma^l_{mn}\frac{\partial Y^m}{\partial x^\alpha}\frac{\partial Y^n}{\partial x^\beta}\right)=0 \tag{4.7}$$

where ${}^{(1)}\Gamma^\gamma_{\alpha\beta}$ (respectively ${}^{(2)}\Gamma^l_{mn}$) denotes the Christoffel symbol of the Levi-Civita connection of g_1 (respectively g_2) and $1\leqslant\alpha, \beta, \gamma\leqslant m_1$; $1\leqslant m, n, l< m_2$. Then, we get the following proposition.

Proposition 4.1. The map $f:(\mathbb{R}^8, N)\to(\mathbb{R}^5, h)$ defined by (4.4) is harmonic map.

Proof. The manifolds $(\mathbb{R}^8, N)$ and $(\mathbb{R}^5, h)$ are flat. Therefore, the Christoffel symbols vanish in (4.7). Now, it is easy to check that the following equations:

$$N^{\alpha\beta}\frac{\partial^2}{\partial x^\alpha\,\partial x^\beta}X_j=0 \qquad 1\leqslant j\leqslant 5$$

hold. This completes the proof.

We now introduce the definition of a harmonic morphism (Eells and Lemaire 1984). Let us consider the differential of the map Y (4.6) at the point x of M_1, i.e. the map $\mathrm{d}Y(x): T_xM_1\to T_{Y(x)}M_2$. Let $\ker(\mathrm{d}Y(x))$ be the kernel of this map. Because of the metric g_1 we are able to define the orthogonal complement $\mathrm{HOR}(x)$ of $\ker(\mathrm{d}Y(x))$ in T_xM_1. Hence, we get the following decomposition:

$$T_xM_1=\ker(\mathrm{d}Y(x))\oplus\mathrm{HOR}(x). \tag{4.8}$$

Then each vector Z of T_xM_1 can be written as follows:

$$Z=Z^{(\mathrm{V})}+Z^{(\mathrm{H})}$$

where $Z^{(\mathrm{V})}\in\ker(\mathrm{d}Y(x))$ is called the vertical part of Z and $Z^{(\mathrm{H})}\in\mathrm{HOR}(x)$ is called the horizontal part of Z. Now, the map Y (4.6) is said to be horizontally conformal if and only if, for each $x\in M_1$, such that $\mathrm{d}Y(x)\neq 0$ and for all Z, $V\in T_xM_1$ we have

$$\langle\mathrm{d}Y(x)Z, \mathrm{d}Y(x)V\rangle_2=k(x)\langle Z^{(\mathrm{H})}, V^{(\mathrm{H})}\rangle_1 \tag{4.9}$$

for some function $k: M_1\to\mathbb{R}$. Here, $\langle\cdot,\cdot\rangle_1$ (respectively $\langle\cdot,\cdot\rangle_2$) denotes the bilinear form on T_xM_1 (respectively $T_{Y(x)}M_2$) induced by g_1 (respectively g_2). Finally, the map Y is called a harmonic morphism if and only if Y is a harmonic and horizontally conformal map. Then we get the following proposition.

Proposition 4.2. The map $f:(\mathbb{R}^8, N)\to(\mathbb{R}^5, h)$ is a harmonic morphism.

Proof. From (4.4) we see that $\ker(\mathrm{d}f(U))$ is generated by the vectors

$$\begin{aligned}
K_1&=-\left(q_1\frac{\partial}{\partial q_0}+q_0\frac{\partial}{\partial q_1}+q_3\frac{\partial}{\partial q_2}+q_2\frac{\partial}{\partial q_3}+r_1\frac{\partial}{\partial r_0}+r_0\frac{\partial}{\partial r_1}+r_3\frac{\partial}{\partial r_2}+r_2\frac{\partial}{\partial r_3}\right)\\
K_2&=\left(-q_2\frac{\partial}{\partial q_0}+q_3\frac{\partial}{\partial q_1}+q_0\frac{\partial}{\partial q_2}-q_1\frac{\partial}{\partial q_3}-r_2\frac{\partial}{\partial r_0}+r_3\frac{\partial}{\partial r_1}+r_0\frac{\partial}{\partial r_2}-r_1\frac{\partial}{\partial r_3}\right)\\
K_3&=\left(-q_3\frac{\partial}{\partial q_0}+q_2\frac{\partial}{\partial q_1}+q_1\frac{\partial}{\partial q_2}-q_0\frac{\partial}{\partial q_3}-r_3\frac{\partial}{\partial r_0}+r_2\frac{\partial}{\partial r_1}+r_1\frac{\partial}{\partial r_2}-r_0\frac{\partial}{\partial r_3}\right).
\end{aligned} \tag{4.10}$$

From this it follows that HOR(U) is generated by the following vectors:

$$
\begin{aligned}
H_1 &= \left(q_1 \frac{\partial}{\partial q_0} + q_1 \frac{\partial}{\partial q_1} + q_2 \frac{\partial}{\partial q_2} + q_3 \frac{\partial}{\partial q_3} - r_0 \frac{\partial}{\partial r_0} - r_1 \frac{\partial}{\partial r_1} - r_2 \frac{\partial}{\partial r_2} - r_3 \frac{\partial}{\partial r_3} \right) \\
H_2 &= \left(r_0 \frac{\partial}{\partial q_0} + r_1 \frac{\partial}{\partial q_1} + r_2 \frac{\partial}{\partial q_2} + r_3 \frac{\partial}{\partial q_3} + q_0 \frac{\partial}{\partial r_0} + q_1 \frac{\partial}{\partial r_1} + q_2 \frac{\partial}{\partial r_2} + q_3 \frac{\partial}{\partial r_3} \right) \\
H_3 &= \left(-r_1 \frac{\partial}{\partial q_0} - r_0 \frac{\partial}{\partial q_1} + r_3 \frac{\partial}{\partial q_2} + r_2 \frac{\partial}{\partial q_3} + q_1 \frac{\partial}{\partial r_0} + q_0 \frac{\partial}{\partial r_1} - q_3 \frac{\partial}{\partial r_2} - q_2 \frac{\partial}{\partial r_3} \right) \qquad (4.11) \\
H_4 &= \left(-r_2 \frac{\partial}{\partial q_0} - r_3 \frac{\partial}{\partial q_i} + r_0 \frac{\partial}{\partial q_2} + r_1 \frac{\partial}{\partial q_3} + q_2 \frac{\partial}{\partial r_0} + q_3 \frac{\partial}{\partial r_1} - q_0 \frac{\partial}{\partial r_2} - q_1 \frac{\partial}{\partial r_3} \right) \\
H_5 &= \left(-r_3 \frac{\partial}{\partial q_0} - r_2 \frac{\partial}{\partial q_1} - r_1 \frac{\partial}{\partial q_2} - r_0 \frac{\partial}{\partial q_3} + q_3 \frac{\partial}{\partial r_0} + q_2 \frac{\partial}{\partial r_1} + q_1 \frac{\partial}{\partial r_2} + q_0 \frac{\partial}{\partial r_3} \right).
\end{aligned}
$$

Let Z and V be arbitrary vectors of $T_U\mathbb{R}^8$. Using (4.10) and (4.11) it is straightforward to check that

$$(\mathrm{d}f(U)Z)^{\mathrm{T}} h(\mathrm{d}f(U)V) = k(U)(Z^{(\mathrm{H})})^{\mathrm{T}} N(V^{(\mathrm{H})}) \qquad (4.12)$$

where $k(U)$ is given by the following equation:

$$k(U) = 4(U^{\mathrm{T}} NU)$$

Comparing (4.9) and (4.12), we see that f is horizontally conformal. From proposition 4.1, f is also harmonic. This completes the proof.

It is worth noting that K_1, K_2 and K_3 are in fact generators of the Lie algebra of the group SL(2, $\mathbb{R}$). Starting from (4.10), we get the following commutation rules:

$$[K_1, K_2] = -2K_3 \qquad [K_2, K_3] = -2K_1 \qquad [K_3, K_1] = 2K_2.$$

Let us endow $H^7(4,4)$ (respectively $H^4(3,2)$) with the metric g' (respectively g''). Then we have the following proposition.

Proposition 4.3. The pseudo-Hopf fibration $\hat{f}: (H^7(4,4), g') \rightarrow (H^4(3,2), g'')$ is a harmonic map.

Proof. Let us first consider a general compact case. Let $\phi: \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a map whose components are harmonic homogeneous polynomials and suppose that $\phi(S^{m-1}) \subseteq S^{n-1}$. Then we know (Smith 1975) that the restriction $\phi': S^{m-1} \rightarrow S^{n-1}$ of ϕ defines a harmonic map. Now let $\Psi: \mathbb{R}^{4p} \rightarrow \mathbb{R}^{2p+1}$ be a map whose components are harmonic homogeneous polynomials and such that $\Psi(H^{4p-1}(2p, 2p)) \subseteq H^{2p}(p+1, p)$. Everthing being algebraically similar to the compact case, the whole proof of Smith can be repeated here word for word. Hence, we get that the restriction $\Psi': H^{4p-1}(2p, 2p) \rightarrow H^{2p}(p+1, p)$ of Ψ defines a harmonic map. Proposition 4.3 happens to be a particular case of this result with $p = 2$.

5. Sigma models on a four-dimensional hyperboloid

Let (M, g) be an arbitrary pseudo-Riemannian manifold endowed with the metric g. We define a smooth field $X: M \rightarrow H^4(3,2): x(x^\mu) \rightarrow X(x) = (X^I(x))$. Using the notations of § 4, we have the following constraint:

$$X(x)^{\mathrm{T}} hX(x) = 1. \qquad (5.1)$$

The sigma model defined on M and with values on $H^4(3,2)$ is a field theory defined by the following Lagrangian density:

$$L(X(x)) = \tfrac{1}{4} g^{\mu\nu} \frac{\partial}{\partial x^\mu} X(x)^{\mathrm{T}} h \frac{\partial}{\partial x^\nu} X(x). \tag{5.2}$$

The Lagrangian density leads to the equations of motion:

$$\Box X^l(x) + g^{\mu\nu} \Gamma^l_{mn} \frac{\partial}{\partial x^\mu} X^m(x) \frac{\partial}{\partial x^\nu} X^n(x) = 0 \qquad 1 \leqslant l \leqslant 5 \tag{5.3}$$

where Γ^l_{mn} is the Christoffel symbol of the Levi-Civita connection on $H^4(3,2)$ and where $\Box$ denotes the Laplace-Beltrami operator associated with M and is given by

$$\Box = 1/\sqrt{|g|} \frac{\partial}{\partial x^\mu}\left(g^{\mu\nu} \sqrt{|g|} \frac{\partial}{\partial x^\nu}\right) \qquad |g| = |\det(g)|. \tag{5.4}$$

From (4.7), (5.3) and (5.4) it follows that X defines a harmonic map. A straightforward computation shows that (5.3) can be written as follows:

$$\Box X^l(x) + g^{\mu\nu}\left(\frac{\partial}{\partial x^\mu} X(x)^{\mathrm{T}} h \frac{\partial}{\partial x^\nu} X(x)\right) X^l(x) = 0. \tag{5.5}$$

We are now going to define equivalent forms of the Lagrangian density (5.2). We start with a smooth field $n : M \to \mathbb{Q}^2 : x \to n(x)$, where $n(x)^{\mathrm{T}} = (q(x) r(x))$, with $q(x) = q_0(x)e_0 + q_1(x)e_1 + q_2(x)e_2 + q_3(x)e_3$ and $r(x) = r_0(x)e_0 + r_1(x)e_1 + r_2(x)e_2 + r_3(x)e_3$ being two smooth Gödel quaternionic fields. Using the notation (3.16) extended to Gödel quaternionic vectors, we introduce the Lagrangian density $L_1(n(x))$ given by

$$L_1(n(x)) = g^{\mu\nu}(D_\mu n(x))^{*} D_\nu n(x) \tag{5.6}$$

with the following constraint:

$$n(x)^{*} n(x) = |q(x)|^2 + |r(x)|^2 = 1. \tag{5.7}$$

Furthermore, we define $D_\mu n(x)$ as the following vector:

$$D_\mu n(x) = \frac{\partial}{\partial x^\mu} n(x) - n(x) A_\mu(x)$$

with

$$A_\mu = \bar{q}(x) \frac{\partial}{\partial x^\mu} q(x) + \bar{r}(x) \frac{\partial}{\partial x^\mu} r(x). \tag{5.8}$$

From (5.7), we immediately check that

$$\bar{A}_\mu(x) = -A_\mu(x).$$

Hence, the 1-form $A(x)$ defined by

$$A(x) = A_\mu(x)\, \mathrm{d}x^\mu$$

can be written

$$A(x) = \omega_1(x)e_1 + \omega_2(x)e_2 + \omega_3(x)e_3$$

where

$$\omega_1 = q_0\, \mathrm{d}q_1 - q_1\, \mathrm{d}q_0 + q_2\, \mathrm{d}q_3 - q_3\, \mathrm{d}q_2 + r_0\, \mathrm{d}r_1 - r_1\, \mathrm{d}r_0 + r_2\, \mathrm{d}r_3 - r_3\, \mathrm{d}r_2 \tag{5.9}$$

$$\omega_2 = q_0\,dq_2 - q_2\,dq_0 + q_1\,dq_3 - q_3\,dq_1 + r_0\,dr_2 - r_2\,dr_0 + r_1\,dr_3 - r_3\,dr_1$$

$$\omega_3 = q_0\,dq_3 - q_3\,dq_0 + q_1\,dq_2 - q_2\,dq_1 + r_0\,dr_3 - r_3\,dr_0 + r_1\,dr_2 - r_2\,dr_1.$$

These forms are in fact related to the vector fields K_1, K_2 and K_3 (4.10) by the following equations:

$$\omega_i(K_j) = -\eta_{ij}$$

where $\eta = \mathrm{diag}(1, -1, 1)$. Let us now consider the transformation given by

$$n(x) \to n'(x) = n(x)s(x) \qquad |s(x)|^2 = 1. \tag{5.10}$$

Under this transformation, $A_\mu(x)$ becomes

$$A'_\mu(x) = \bar{s}(x)\frac{\partial}{\partial x^\mu}s(x) + \bar{s}(x)A_\mu(x)s(x). \tag{5.11}$$

From (5.10) and (5.11) we get

$$D'_\mu n'(x) = (D_\mu n(x))s(x). \tag{5.12}$$

Furthermore, from § 2 and (5.12), we easily check that $L_1(n(x))$ is a $\mathrm{SL}(2,\mathbb{R})$-invariant Lagrangian density. Finally, (5.10)-(5.12) show that (5.6) defines a gauge field theory with gauge field $A_\mu(x)$ and gauge group $\mathrm{SL}(2,\mathbb{R})$. We now define the vector $U(x)$ such that $U(x)^{\mathrm{T}} = (q_0(x)q_1(x)q_2(x)q_3(x)r_0(x)r_1(x)r_2(x)r_3(x))$. Using the above defined notations, we set

$$X(x) = f(U(x)). \tag{5.13}$$

According to (4.5) and (5.7), we have

$$X(x)^{\mathrm{T}}hX(x) = (U(x)^{\mathrm{T}}NU(x))^2 = 1.$$

Therefore, (5.13) becomes $X(x) = \hat{f}(U(x))$. Let us consider the Lagrangian density $L(X(x))$ associated with the field $X(x)$ and defined by (5.2). Then we get the following result.

Proposition 5.1. We have

$$L(X(x)) = L_1(n(x)) \tag{5.14}$$

i.e. the pseudo-Hopf fibration $\hat{f}$ reduces (5.6) to (5.2).

Proof. Using (5.6) and (5.8) we can write

$$L_1(n(x)) = g^{\mu\nu}\left\{\frac{\partial\bar{q}}{\partial x^\mu}\frac{\partial q}{\partial x^\nu} + \frac{\partial\bar{r}}{\partial x^\mu}\frac{\partial r}{\partial x^\nu} + \left(\bar{q}\frac{\partial q}{\partial x^\mu} + \bar{r}\frac{\partial r}{\partial x^\mu}\right)\left(\bar{q}\frac{\partial q}{\partial x^\nu} + \bar{r}\frac{\partial r}{\partial x^\nu}\right)\right\}. \tag{5.15}$$

Starting from (4.4) and (5.15), we get (5.14) after a long but easy computation.

From the field $X(x)$ (5.13) such that $X(x)^{\mathrm{T}} = (X_1(x), \ldots, X_5(x))$, we introduce the Gödel quaternionic field $W: M \to \mathbb{Q}: x \to W(x) = W_0(x)e_0 + W_1(x)e_1 + W_2(x)e_2 + W_3(x)e_3$, where

$$W_k(x) = X_{k+2}(x)/(1 + X_1(x)) \qquad 0 \leqslant k \leqslant 3 \qquad X_1(x) \neq -1. \tag{5.16}$$

In fact, (5.16) defines the stereographic projection of the manifold $H = \{(X_1, \ldots, X_5) \in H^4(3,2)\colon X_1 \neq -1\}$ to the hyperplane $X_1 = 1$ (endowed with the metric $\mathrm{diag}(1, -1, 1, -1)$) with centre $(X_1, \ldots, X_5) = (-1, 0, \ldots, 0)$. By a straightforward computation we get $L(X(x)) = L_2(W(x))$ where

$$L_2(W(x)) = g^{\mu\nu}(1 + |W(x)|^2)^{-2} \frac{\partial}{\partial x^\mu} \bar{W}(x) \frac{\partial}{\partial x^\nu} W(x). \tag{5.17}$$

Now let $P(W(x))$ be the matrix field defined by

$$P(W(x)) = (1 + |W(x)|^2)^{-1} \begin{bmatrix} 1 & \bar{W}(x) \\ W(x) & |W(x)|^2 \end{bmatrix} \qquad |W(x)|^2 \neq -1. \tag{5.18}$$

We introduce the following Lagrangian density:

$$L_3(P(W(x))) = \tfrac{1}{2} \operatorname{Tr}\left(g^{\mu\nu} \frac{\partial}{\partial x^\mu} P \frac{\partial}{\partial x^\nu} P \right). \tag{5.19}$$

Then we get the following proposition.

Proposition 5.2. We have

$$L(X(x)) = L_3(P(W(x))). \tag{5.20}$$

Proof. Using (5.18) and (5.19) we get $L_3(P(W(x))) = L_2(W(x))$. We know that $L_2(W(x)) = L(X(x))$. This completes the proof.

The geometrical meaning of (5.20) is given by the following proposition.

Proposition 5.3. The Gödel quaternionic field $P(W(x))$ parametrises a point on the hyperboloid $H^4(3,2)$.

Proof. From (5.18) we see that the equations

$$P(W(x))^2 = P(W(x)) \qquad P(W(x))^* = P(W(x)) \qquad \operatorname{Tr}(P(W(x))) = 1$$

hold. This means that $P(W(x))$ is a projector on a one-dimensional submodule of $\mathbb{Q}^2$. Referring to the usual quaternionic case (Berger 1987), we see that $P(W(x))$ lives on the coset

$$U_0^{(*)}(2, \mathbb{Q}) / U_0^{(*)}(1, \mathbb{Q}) \times U_0^{(*)}(1, \mathbb{Q}) \tag{5.21}$$

where we have used the notation (3.18). Using (3.15) and (3.20), (5.21) becomes

$$\mathrm{Sp}(2, \mathbb{R}) / \mathrm{Sp}(1, \mathbb{R}) \times \mathrm{Sp}(1, \mathbb{R}) \tag{5.22}$$

which is a pseudo-Riemannian symmetric space (Berger 1957). Now, according to the following results (Barut and Raczka 1986),

$$\mathrm{Sp}(1, \mathbb{R}) \simeq \mathrm{SL}(2, \mathbb{R}) \qquad \mathrm{Sp}(2, \mathbb{R}) \simeq \mathrm{SO}(3, 2)$$

and

$$\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \simeq \mathrm{SO}(2, 2)$$

the coset (5.22) can be identified with $\mathrm{SO}(3,2)/\mathrm{SO}(2,2)$, which is nothing other than the hyperboloid $H^4(3,2)$ (Wolf 1974). Finally, it is worth writing the field equation associated with (5.19). From (5.5), (5.16) and (5.18) we get

$$[P(W(x)), \Box P(W(x))] = 0. \tag{5.23}$$

Lagrangian densities (5.17) and (5.19) happen to be generalisations of Lagrangian densities either of the $\mathbb{C}P^1$ sigma model (Gürsey and Tze 1980) or of the ΩP^1 sigma model (Lambert and Piette 1988). Therefore (5.21) can be seen as the projective quaternionic space $\mathbb{Q}P^1$. Though the non-commutative ring $\mathbb{Q}$ involves zero divisors, we have proved elsewhere that it is possible to define projective spaces $\mathbb{Q}P^n$ (Lambert 1988).

6. Explicit solutions

A general method for solving equation (5.23) is not known. We simply give here three examples of particular solutions.

(i) Let M be the hyperboloid $H^4(3,2)$ endowed with the metric g defined as follows:

$$g(x) = h'/(1+x^{\mathrm{T}}h'x)^2 \qquad x^{\mathrm{T}}h'x \neq -1$$

where $h' = \mathrm{diag}(1,-1,1,-1)$ and $x^{\mathrm{T}} = (x_0\, x_1\, x_2\, x_3)$. We introduce the Gödel quaternionic field $W: H^4(3,2) \to \mathbb{Q} : x \to W(x) = x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3$. It is now easy to check that $P(W(x))$ is a solution of the equation (5.23). This solution happens to be the non-compact generalisation of the 'instanton' of the sigma model defined on S^4 and with values on the usual quaternionic projective space $\mathbb{H}P^1$ (Gava *et al* 1979, Fujii 1985).

(ii) Let us consider a particular metric g on the hyperboloid $H^4(3,2) = M$. We introduce the harmonic homogeneous polynomial $P(x) = P(x_0, \ldots, x_4)$ of degree three given by

$$P(x_0,\ldots,x_4) = \tfrac{1}{3}(x_0^3 - 3x_0x_1^2 - \tfrac{3}{2}x_0(x_2^2 + 2x_3^2 + x_4^2) + \tfrac{3}{2}\sqrt{3}x_1(x_2^2 - x_4^2) + 3\sqrt{3}x_2x_3x_4). \tag{6.1}$$

It is then possible to define the field $X: \mathbb{R}^5 \to \mathbb{R}^5$ given by

$$X(x) = \nabla P(x). \tag{6.2}$$

From (6.1), we easily check that $P(x)$ satisfies the following equation:

$$\nabla P(x)^{\mathrm{T}} h \nabla P(x) = (x^{\mathrm{T}}hx)^2. \tag{6.3}$$

Polynomials satisfying (6.3) are called 'eikonals' in the language of of geometrical optics. From (6.2), we see that the components of $X(x)$ are harmonic homogeneous polynomials of degree two. Furthermore, from (6.3) it follows that $X(H^4(3,2)) \subseteq H^4(3,2)$. By a straightforward computation, it is now possible to show that the restriction $\hat{X}: H^4(3,2) \to H^4(3,2)$ of the field X defines a harmonic map (i.e. satisfies equation (5.5)). Using the stereographic projection (5.16), we define the Gödel quaternionic field $\hat{W}: H^4(3,2) \to \mathbb{Q} : x \to \hat{W}(x) = \hat{W}_0(x)e_0 + \hat{W}_1(x)e_1 + \hat{W}_2(x)e_2 + \hat{W}_3(x)e_3$ such that

$$\hat{W}_k(x) = \hat{X}_{k+2}(x)/(1+\hat{X}_1(x)) \qquad 0<k<3 \qquad \hat{X}_1(x) \neq -1.$$

Using an explicit form for the metric g, we finally check that $P(\hat{W}(x))$, defined from (5.18), is a solution of (5.23).

(iii) Let M be the four-dimensional manifold endowed with the following metric:

$$g(x) = h'/(x^{\mathrm{T}}h'x) \qquad x^{\mathrm{T}}h'x > 0$$

where $x^{\mathrm{T}} = (x_0x_1x_2x_3)$. We define the Gödel quaternionic field $W: M \to \mathbb{Q}$ as follows:

$$W(x) = (x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3)/(x^{\mathrm{T}}h'x)^{1/2}.$$

Referring to (5.18), we define the projector $P(W(x))$. A straightforward computation shows that $P(W(x))$ is a solution of equation (5.23). According to the language of Yang-Mills theories we call this solution a generalised 'meron' (de Alfaro *et al* 1976).

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